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Addendum to "Field Theory of the Two-Dimensional Ising Model: Equivalence to the Free Particle One-Dimensional Dirac Equation"

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An essential step in an earlier work[1] is the disentanglement of the small operator $\mathcal{O}$ from the large operator $2Jn$ in the expression $M = e^{2Jn+\mathcal{O}}$. Substituting Eq. (43) into Eq. (39) of Ref. 1, we can write the latter in the form

$$e^{-Jn}Me^{-Jn} = e^{-Jn}e^{2Jn+\mathcal{O}}e^{-Jn} = 1 + [(\sinh 2J)/2J]\mathcal{O} \tag{69}$$

If $\mathcal{O}$ commuted with n, we would expect to obtain

$$e^{\mathcal{O}} \cong 1 + \mathcal{O} \tag{70}$$

to first order in $\mathcal{O}$. Thus the effect of the lack of commutativity is to introduce the correction factor $(\sinh 2J)/2J$. For an isotropic lattice ($\sinh 2J = 1$) the correction factor amounts to 1.13, and ignoring it would lead to a 13% error. This kind of problem is familiar in the theory of Lie algebras[2,3] and we will

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give here an alternative derivation of the correction factor as an application of the well-known Campbell[4]–Baker[5]–Hausdorff[6] (CBH) formula. Let $A = n$, $B = (1/2J)\mathcal{O}$, $\lambda = 2J$, and

$$F(\lambda) = e^{-\lambda A/2} e^{\lambda(A+B)} e^{-\lambda A/2} \tag{71}$$

From the general CBH formula we know that the general term in the Taylor expansion of $F(\lambda)$ can be expressed in terms of multiple commutators of A and B. But the fact that we are working only to the first order in B brings about a simplification which is most evident in the first derivative:

$$\begin{aligned} F'(\lambda) &= -\tfrac{1}{2} e^{-\lambda A/2} A e^{\lambda(A+B)} e^{-\lambda A/2} - \tfrac{1}{2} e^{-\lambda A/2} e^{\lambda(A+B)} A e^{-\lambda A/2} \\ &\quad + \tfrac{1}{2} e^{-\lambda A/2} (A + B) e^{\lambda(A+B)} e^{-\lambda A/2} + \tfrac{1}{2} e^{-\lambda A/2} e^{\lambda(A+B)} (A + B) e^{-\lambda A/2} \\ &= \tfrac{1}{2} e^{-\lambda A/2} B e^{\lambda(A+B)} e^{-\lambda A/2} + \tfrac{1}{2} e^{-\lambda A/2} e^{\lambda(A+B)} B e^{-\lambda A/2} \\ &\cong \tfrac{1}{2} e^{-\lambda A/2} B e^{\lambda A/2} + \tfrac{1}{2} e^{\lambda A/2} B e^{-\lambda A/2} \end{aligned} \tag{72}$$

This shows that $F'(\lambda)$ is an even function of λ, so that only even terms appear in its Taylor series,

$$F'(\lambda) = F'(0) + (\lambda^2/2!)F'''(0) + (\lambda^4/4!)F^{v}(0) + \cdots \tag{73}$$

with the coefficients given by

$$F'(0) = B \tag{74a}$$

$$F'''(0) = \tfrac{1}{4}[A, [A, B]] \tag{74b}$$

$$F^{v}(0) = \tfrac{1}{16}[A, [A, [A, [A, B]]]] \tag{74c}$$

and so forth. Now a further simplification enters because of the special relationship of the operators A and B. B creates or annihilates pairs, so that each commutation with A (the occupation number) gives the factor ± 2. Consequently all of the odd derivations are equal,

$$F'''(0) = F^{v}(0) = \cdots = B \tag{75}$$

and as a result of this drastic simplification Eq. (73) can be summed to

$$F'(\lambda) = B \cosh \lambda \tag{76}$$

which can also be obtained directly from Eq. (72). Integrating with respect to λ, we find

$$\begin{aligned} F(2J) &= F(0) + \int_0^{2J} F'(\lambda)\, d\lambda = 1 + B \int_0^{2J} \cosh \lambda \, d\lambda \\ &= 1 + B \sinh 2J = 1 + [(\sinh 2J)/2J]\mathcal{O} \end{aligned} \tag{77}$$

with the correction factor identical to that in Eq. (69).

In conclusion, the author wishes to thank Prof. C.-H. Woo for a provocative comment which prompted this alternative derivation.

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